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Conservation laws of scaling-invariant field equations

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Abstract
A simple conservation law formula for field equations with a scaling symmetry is presented. The formula uses adjoint-symmetries of the given field equation and directly generates all local conservation laws for any conserved quantities having non-zero scaling weight. Applications to several soliton equations, fluid flow and nonlinear wave equations, Yang–Mills equations and the Einstein gravitational field equations are considered.

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1. Introduction

Conservation laws are central to the analysis of physical field equations by providing conserved quantities, such as energy, momentum and angular momentum. For a given field equation, local conservation laws are well known to arise through multipliers [1], analogous to integrating factors of ODEs [2], with the product of the multiplier and the field equation being a total divergence expression. Such divergences correspond to a conserved current vector for solutions of the field equation whenever the multiplier is non-singular. If a field equation possesses a Lagrangian, Noether's theorem [1] shows that the multipliers for local conservation laws consist of symmetries of the field equation such that the action principle is invariant (to within a boundary term). Moreover, the variational relation between the Lagrangian and the field equation yields an explicit formula for the resulting conserved current vector. This characterization of multipliers for a Lagrangian field equation has a generalization to any field equation by means of adjoint-symmetries [3, 4], whether or not a Lagrangian formulation exists.

Recall, geometrically, symmetries are tangent vector fields on the solution space of a field equation and thus are determined as field variations satisfying the linearization of the field equation on its entire solution space. Adjoint-symmetries are defined to satisfy the adjoint equation of the symmetry determining equation on the solution space of a field equation[1]. (As such, unlike for symmetries, there is no obvious geometrical motion or invariance associated with adjoint-symmetries.) Through standard results in the calculus of variations [1], it is known that the multipliers for local conservation laws are precisely adjoint-symmetries of the field equation subject to a certain adjoint invariance condition[2] [3]. This allows a system of determining equations for multipliers to be formulated in terms of the adjoint-symmetry determining equation augmented by extra determining equations[3] [4]. In addition, the resulting conservation laws are yielded by means of a homotopy integral expression [1, 3, 4] involving just the field equation and the multiplier, which is derived from the adjoint invariance condition, analogously to the line integral formula for first integrals of ODEs [2].

The purpose of this paper is to show that in the physically interesting situation where a field equation possesses a scaling symmetry, then the adjoint invariance condition and homotopy integral formula can be completely by-passed for obtaining conservation laws. In particular, a simple algebraic formula that directly generates conservation laws in terms of adjoint-symmetries for any such field equation is presented. Most importantly, when applied to a multiplier, the formula recovers the corresponding conservation law determined by the multiplier, to within a proportionality factor. This factor turns out to be the scaling weight of the conserved quantity defined from the conservation law. Consequently, all conserved quantities with non-zero scaling weight are obtainable from this formula[4].

In section 2, the conservation law formula is derived. Examples and applications of this formula are presented in section 3. As new results, first, a recursion formula is obtained for the local higher order conservation laws of the sine–Gordon equation and a vector generalization of the Korteweg–de Vries equation; second, a simple proof is given for closing a gap in the classification of local conservation laws of the Yang–Mills equations and Einstein gravity equations. Some concluding remarks are made in section 4.

2. Conservation law formula

Consider a general system of field equations

$$\Upsilon^A(x, u, \partial u, \partial^2 u, \ldots) = 0 \tag{2.1}$$

for field variables $u^a(x)$ depending on a total of $n \geqslant 2$ time and space variables x^α, with $\partial^k u$ denoting partial derivatives $u^a{}_{,\alpha_1\cdots\alpha_k} = \partial^k u^a(x)/\partial x^{\alpha_1}\cdots\partial x^{\alpha_k}$, up to some finite differential order. (The coordinate indices α, β, γ run 0 to $n-1$; the field index a runs 1 to N; the equation index A runs 1 to m. Summation is assumed over any repeated indices. This formalism allows the number of components of the fields u^a and equations Υ^A to be different.) For simplicity of presentation, the differential order of system (2.1) will be restricted to $k \leqslant 2$.

Symmetries of the field equations (2.1) are the solutions $\delta u^a = \eta^a(x, u, \partial u, \partial^2 u, \ldots)$ of the linearized equations

$$\mathcal{L}_\Upsilon(\eta)^A = \eta^a \Upsilon^A{}_{,a} + (D_\alpha \eta^a)\Upsilon^A{}_{,a}^{\ \alpha} + (D_\alpha D_\beta \eta^a)\Upsilon^A{}_{,a}^{\ \alpha\beta} = 0 \tag{2.2}$$

[1] Throughout, 'symmetries' will refer to local (point or generalized) symmetries in evolutionary form (see [1]). A symmetry, or adjoint-symmetry, is trivial if it vanishes on the solution space of the field equations. Two symmetries, or adjoint-symmetries, are considered equivalent if they differ by one that is trivial.

[2] The adjoint invariance condition holds for only certain adjoint-symmetries, if any, in an equivalence class.

[3] The determining equation for adjoint-symmetries is the same as that for symmetries when and only when a field equation is self-adjoint, which is also the necessary and sufficient condition for the field equation to have a Lagrangian formulation. In this case, adjoint-symmetries are symmetries, and the adjoint invariance condition is equivalent to invariance of the action principle.

[4] For a linear field equation, this formula yields all local conservation laws whenever, as is typically the case, the corresponding multipliers have non-negative scaling weight under a scaling purely on the fields, as considered in [5].

for all $u^a(x)$ satisfying system (2.1), where D_α is the total derivative with respect to x^α, and where $\Upsilon^A{}_{,a}$, $\Upsilon^A{}_{,a}^{\ \alpha}$, etc denote partial derivatives $\partial\Upsilon^A/\partial u^a$, $\partial\Upsilon^A/\partial u^a{}_{,\alpha}$, etc. The adjoint of equation (2.2) is given by

$$\mathcal{L}^*_\Upsilon(\omega)_a = \omega_A \Upsilon^A{}_{,a} - D_\alpha\big(\omega_A \Upsilon^A{}_{,a}^{\ \alpha}\big) + D_\alpha D_\beta\big(\omega_A \Upsilon^A{}_{,a}^{\ \alpha\beta}\big) = 0 \tag{2.3}$$

whose solutions $\omega_A(x, u, \partial u, \partial^2 u, \ldots)$ for all $u^a(x)$ satisfying system (2.1) are the adjoint-symmetries of the field equations (2.1). Note the operators $\mathcal{L}_\Upsilon$ and $\mathcal{L}^*_\Upsilon$ are related by the identity

$$\omega_A \mathcal{L}_\Upsilon(\eta)^A - \eta^a \mathcal{L}^*_\Upsilon(\omega)_a = D_\alpha \Phi^\alpha(\omega, \eta; \Upsilon) \tag{2.4}$$

with

$$\Phi^\alpha(\omega, \eta; \Upsilon) = \eta^a\big(\omega_A \Upsilon^A{}_{,a}^{\ \alpha} - D_\beta\big(\omega_A \Upsilon^A{}_{,a}^{\ \alpha\beta}\big)\big) - (D_\beta \eta^a)\omega_A \Upsilon^A{}_{,a}^{\ \alpha\beta}. \tag{2.5}$$

Hence, this expression (2.5) yields a local conservation law $D_\alpha \Phi^\alpha(\omega, \eta) = 0$ for any pair ω_A, η^a, on all solutions of the field equations (2.1).

Now suppose the field equations are invariant under a scaling of the variables

$$x^\alpha \to \lambda^{p^{(\alpha)}} x^\alpha \qquad u^a \to \lambda^{q^{(a)}} u^a \tag{2.6}$$

with $p^{(\alpha)} = \text{const}$, $q^{(a)} = \text{const}$. From the corresponding scaling symmetry, given by

$$\delta_s u^a = \eta_s{}^a(x, u, \partial u) = q^{(a)} u^a - p^{(\alpha)} x^\alpha u^a{}_{,\alpha} \tag{2.7}$$

the expression $\Phi^\alpha(\omega, \eta_s)$ produces a conserved current in terms of any adjoint-symmetry ω_A.

Proposition 2.1. *For scaling invariant field equations (2.1), every adjoint-symmetry (2.3) generates a conserved current on all solutions of (2.1) by the formula*

$$\begin{aligned}\Phi^\alpha_\omega = {} & \big(q^{(a)} u^a - p^{(\gamma)} x^\gamma u^a{}_{,\gamma}\big)\big(\omega_A \Upsilon^A{}_{,a}^{\ ;\alpha} - D_\beta\big(\omega_A \Upsilon^A{}_{,a}^{\ ,\alpha\beta}\big)\big) \\ & + \big((p^{(\beta)} - q^{(a)}) u^a{}_{,\beta} + p^{(\gamma)} x^\gamma u^a{}_{,\beta\gamma}\big)\omega_A \Upsilon^A{}_{,a}^{\ \alpha\beta}.\end{aligned} \tag{2.8}$$

Consider, now, a multiplier $Q_A(x, u, \partial u, \partial^2 u, \ldots)$ for a local conservation law

$$Q_A \Upsilon^A = D_\alpha \Psi^\alpha_Q \tag{2.9}$$

that is assumed to be homogeneous[5] under the scaling symmetry, so

$$\delta_s Q_A = \mathcal{L}_Q(\eta_s)_A = r_{(A)} Q_A - p^{(\alpha)} x^\alpha D_\alpha Q_A \tag{2.10}$$

with scaling weight $r_{(A)} = \text{const}$. Let $s^{(A)} = \text{const}$ be the scaling weight of the field equations,

$$\delta_s \Upsilon^A = \mathcal{L}_\Upsilon(\eta_s)^A = s^{(A)} \Upsilon^A - p^{(\alpha)} x^\alpha D_\alpha \Upsilon^A. \tag{2.11}$$

Due to the scaling homogeneity of Ψ^α_Q, the constant $r_{(A)} + s^{(A)}$ is independent of the index A. Then, the following important relation holds between the conserved currents Ψ^α_Q and Φ^α_Q.

Theorem 2.2. *In terms of the scaling weights of the field equations (2.11) and the multiplier (2.10), every local conservation law (2.9) satisfies the scaling relation*

$$\Phi^\alpha_Q \simeq w_Q \Psi^\alpha_Q \qquad w_Q = r_{(A)} + s^{(A)} + \sum_\alpha p^{(\alpha)} \tag{2.12}$$

for all $u^a(x)$ satisfying the field equations, where $\simeq$ denotes equality to within a trivial conserved current[6] *$D_\beta \Theta^{\alpha\beta}$ for some local expression $\Theta^{\alpha\beta} = -\Theta^{\beta\alpha}$. Moreover, w_Q is simply*

[5] This entails no essential loss of generality, as scaling invariance of the field equations implies that any multiplier is, formally, a Laurent series of homogeneous multipliers.

[6] A conservation law is trivial if, on the solution space of the field equations, it has the form of the divergence of an antisymmetric tensor. Two conservation laws are equivalent if they differ by a trivial conservation law.

the scaling weight of the flux integral $\int \Psi^{\alpha}_{Q} n_{\alpha}\, \mathrm{d}^{n-1}x$ *defined on any* $(n-1)$*-dimensional hypersurface* $x^{\alpha} n_{\alpha} = \text{const}$ *(with normal vector* n_{α}*),*

$$\delta_{\mathrm{s}} \int \Psi^{\alpha}_{Q} n_{\alpha}\, \mathrm{d}^{n-1}x = w_{Q} \int \Psi^{\alpha}_{Q} n_{\alpha}\, \mathrm{d}^{n-1}x. \tag{2.13}$$

Definition 2.3. *A conservation law (2.9) will be called (non)critical with respect to the scaling (2.6) if the scaling weight (2.13) of the corresponding conserved quantity is (non)zero.*

Corollary 2.4. *As all multipliers necessarily are given by adjoint-symmetries, the conservation law formula (2.8) consequently generates all noncritical conservation laws of the field equations (2.1).*

The proof of relation (2.12) starts from the identity (2.4) with $\eta^{a} = \eta_{\mathrm{s}}{}^{a}$. We substitute the condition $\mathcal{L}^{*}_{\omega}(\Upsilon)_{a} = -\mathcal{L}^{*}_{\Upsilon}(\omega)_{a}$ on ω_{A} (holding for all $u^{a}(x)$ without use of the field equations), which is necessary and sufficient [1] for an adjoint-symmetry to be a multiplier $Q_{A} = \omega_{A}$. Here, $\mathcal{L}^{*}_{\omega}$ is the adjoint of the linearization operator $\mathcal{L}_{\omega}$ defined analogously to $\mathcal{L}^{*}_{\Upsilon}$ and $\mathcal{L}_{\Upsilon}$. We next use the adjoint relation

$$\eta_{\mathrm{s}}{}^{a} \mathcal{L}^{*}_{\omega}(\Upsilon)_{a} = \Upsilon^{A} \mathcal{L}_{\omega}(\eta_{\mathrm{s}})_{A} - D_{\alpha} \Psi^{\alpha}(\Upsilon, \eta_{\mathrm{s}}; \omega). \tag{2.14}$$

Substituting the scaling relations (2.10) and (2.11), followed by integrating by parts, we obtain

$$w_{Q} Q_{A} \Upsilon^{A} = D_{\alpha}\big(\Phi^{\alpha}_{Q} + \Psi^{\alpha}(\Upsilon, \eta_{\mathrm{s}}; Q)\big) \tag{2.15}$$

where, note, the last term in this divergence vanishes when $\Upsilon^{A} = 0$. Then conservation law equation (2.9) leads to the scaling relation (2.12).

3. Examples and applications

3.1. Soliton equations

For applications of the main conservation law formulae (2.8) and (2.12), consider, firstly, soliton field equations in 1 + 1 dimensions.

3.1.1. Korteweg–de Vries equation. The KdV equation in physical form for scalar field $u(t, x)$ is given by

$$\Upsilon(u, u_{t}, u_{x}, u_{xxx}) = u_{t} + u u_{x} + u_{xxx} = 0. \tag{3.1}$$

This field equation is invariant under the scaling $t \to \lambda^{3} t, x \to \lambda x, u \to \lambda^{-2} u$. The corresponding scaling symmetry

$$\delta u = \eta_{\mathrm{s}} = -2u - 3t u_{t} - x u_{x} \tag{3.2}$$

is a solution of the linearized field equation

$$\mathcal{L}_{\Upsilon}(\eta) = D_{t}\eta + u_{x}\eta + u D_{x}\eta + D^{3}_{x}\eta = 0 \tag{3.3}$$

for all $u(t, x)$ satisfying the KdV equation. The adjoint of equation (3.3) is given by

$$\mathcal{L}^{*}_{\Upsilon}(\omega) = -D_{t}\omega - u D_{x}\omega - D^{3}_{x}\omega = 0 \tag{3.4}$$

whose solutions ω for all $u(t, x)$ satisfying (3.1) are the adjoint-symmetries of the KdV equation. Here, we see $\mathcal{L}^{*}_{\Upsilon} \neq \mathcal{L}_{\Upsilon}$, reflecting the fact that the KdV equation (3.1) lacks a local Lagrangian formulation in terms of $u(t, x)$. Now, for any adjoint-symmetry ω, the conservation law formula (2.8) gives the conserved density

$$\Phi^{t}_{\omega} = -(3t u_{t} + x u_{x} + 2u)\omega. \tag{3.5}$$

From the obvious solution $w = u$ of equation (3.4), we consider the infinite sequence of KdV adjoint-symmetries $\omega_{(k)} = (\mathcal{R}^*)^k u$, $k = 0, 1, 2, \ldots$, generated by the operator

$$\mathcal{R}^* = D_x^2 + \tfrac{1}{3}u + \tfrac{1}{3}D_x^{-1}(uD_x) \tag{3.6}$$

which is the adjoint of the well-known KdV recursion operator [6]

$$\mathcal{R} = D_x^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D_x^{-1}. \tag{3.7}$$

Note that, under the KdV scaling symmetry, $\omega_{(k)} \to \lambda^{-2(1+k)}\omega_{(k)}$.

Each adjoint-symmetry $\omega_{(k)}$ is known to be a multiplier for a local conservation law of the form

$$D_t\Psi^t_{(k)}(u, u_x, u_{xx}, \ldots) + D_x\Psi^x_{(k)}(u, u_x, u_{xx}, \ldots) = 0 \tag{3.8}$$

on KdV solutions $u(t, x)$, through lengthy calculations. For instance, originally the KdV conservation laws were derived one by one via the Miura transformation [7], from which the multipliers can be calculated. An alternative approach has involved extracting the conserved densities one by one through a residue method using a formal symmetry (pseudo-differential operator) [1] for the KdV equation. More recently, in the conserved densities were obtained one at a time from a homotopy integral formula in terms of the adjoint-symmetries, after a verification of the adjoint invariance condition on each one [3].

Here, by-passing such cumbersome steps, formula (3.5) yields the resulting conserved densities directly in terms of $\omega_{(k)}$,

$$\Phi^t_\omega \simeq D_x^{-1}(u_x\omega_{(k)}) - 2u\omega_{(k)} + 3t(uu_x + u_{xxx})\omega_{(k)} \simeq -3\mathcal{R}^*(\omega_{(k)}) \tag{3.9}$$

which follows by means of properties of $\mathcal{R}^*$. This leads to a simple explicit recursion formula for all of the KdV local conservation laws

$$\Psi^t_{(k)} = \frac{3}{3+2k}\omega_{(k+1)} = \frac{1+2k}{3+2k}\mathcal{R}^*\big(\Psi^t_{(k-1)}\big) \tag{3.10}$$

to within a trivial conserved density $D_x\Theta$, as a result of the scaling formula (2.12).

3.1.2. Sine–Gordon equation. The sine–Gordon equation is given by

$$\Upsilon(u, u_{tx}) = u_{tx} - \sin u = 0 \tag{3.11}$$

for scalar field $u(t, x)$. This is a Lagrangian field equation with the scaling invariance $t \to \lambda^{-1}t$, $x \to \lambda x$, $u \to u$. The corresponding scaling symmetry is $\delta u = \eta_{\mathrm{s}} = tu_t - xu_x$ which is a solution of the linearized field equation

$$\mathcal{L}_\Upsilon(\eta) = D_t D_x\eta - (\cos u)\eta = 0 \tag{3.12}$$

for all $u(t, x)$ satisfying the sine–Gordon equation. Here, as we have $\mathcal{L}^*_\Upsilon = \mathcal{L}_\Upsilon$, symmetries are the same as adjoint-symmetries, $\omega = \eta$. Hence, for any symmetry η, the conservation law formula (2.8) gives the conserved density

$$\Phi^t_\eta = (xu_x - tu_t)D_x\eta. \tag{3.13}$$

We now consider the well-known infinite sequence of symmetries $\eta^{(k)} = (\mathcal{R})^k u_x$, $k = 0, 1, 2, \ldots$, generated by the sine–Gordon recursion operator [6]

$$\mathcal{R} = D_x^2 + u_x D_x^{-1}(u_x D_x) \tag{3.14}$$

starting from the translation symmetry $\eta = u_x$. Under the sine–Gordon scaling, note $\eta^{(k)} \to \lambda^{-2k}\eta^{(k)}$. Each symmetry $\eta^{(k)}$ is a multiplier for a local conservation law of the form (3.8) on sine–Gordon solutions $u(t, x)$. These conservation laws were found originally by an application of Noether's theorem [8] and subsequently were derived by the same techniques used for the KdV equation [1], yielding the conserved densities one at a time through lengthy calculations. Here, similarly to the KdV case, formula (3.13) directly leads instead to a simple explicit expression for all of the sine–Gordon local conservation laws,

$$\Phi^t_\eta \simeq -D_x^{-1}(u_x D_x \eta^{(k)}) = -\mathcal{R}_{\text{nonloc}}(\eta^{(k)})/u_x = \Phi^t_{(k)} \tag{3.15}$$

and hence

$$\Psi^t_{(k)} \simeq \frac{-1}{1+2k}\Phi^t_{(k)} = \frac{1}{1+2k}\big(\eta^{(k+1)} - D_x^2\eta^{(k)}\big)\big/u_x \tag{3.16}$$

from the scaling formula (2.12). Here $\mathcal{R}_{\text{nonloc}}$ stands for the nonlocal part of $\mathcal{R}$. If the relation $-\big(D_x\Phi^t_{(k)}\big)\big/u_x = D_x\eta^{(k)}$ is substituted into expression (3.16) to get

$$\eta^{(k+1)} = -u_x\Phi^t_{(k)} - D_x\left(\frac{1}{u_x}D_x\Phi^t_{(k)}\right) \tag{3.17}$$

then equation (3.15) yields an explicit conservation law recursion formula

$$\Psi^t_{(k)} = \frac{2k-1}{2k+1}\hat{\mathcal{R}}\big(\Psi^t_{(k-1)}\big) \tag{3.18}$$

with

$$\hat{\mathcal{R}} = {u_x}^2 D_x \frac{1}{{u_x}^2} D_x + \frac{1}{2}{u_x}^2 + D_x^{-1}\left(\left(\frac{u_{xxx}}{u_x} + \frac{1}{2}{u_x}^2\right)D_x\right) \tag{3.19}$$

representing a recursion operator on conserved densities.

3.1.3. Modified Korteweg–de Vries vector equation. It is known that the recursion operator and symmetry hierarchy of the sine–Gordon equation are closely related to that of the modified Korteweg–de Vries equation $u_t + \frac{3}{2}u^2u_x + u_{xxx} = 0$. This scalar field equation has an interesting generalization [9]

$$\boldsymbol{\Upsilon}(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{u}_x, \boldsymbol{u}_{xxx}) = \boldsymbol{u}_t + \tfrac{3}{2}\boldsymbol{u}\cdot\boldsymbol{u}\,\boldsymbol{u}_x + \boldsymbol{u}_{xxx} = 0 \tag{3.20}$$

for an N-dimensional vector field $\boldsymbol{u}(t,x)$, with any $N \geqslant 1$. The vector mKdV equation (3.20) is invariant under the scaling $t \to \lambda^3 t$, $x \to \lambda x$, $\boldsymbol{u} \to \lambda^{-1}\boldsymbol{u}$. Its symmetries are the solutions of the linearized field equation

$$\mathcal{L}_{\boldsymbol{\Upsilon}}(\boldsymbol{\eta}) = D_t\boldsymbol{\eta} + 3\boldsymbol{u}\cdot\boldsymbol{\eta}\,\boldsymbol{u}_x + \tfrac{3}{2}\boldsymbol{u}\cdot\boldsymbol{u}D_x\boldsymbol{\eta} + D_x^3\boldsymbol{\eta} = 0 \tag{3.21}$$

for all mKdV solutions $\boldsymbol{u}(t,x)$, while its adjoint-symmetries are the solutions of the adjoint of equation (3.21)

$$\mathcal{L}^*_{\boldsymbol{\Upsilon}}(\boldsymbol{\omega}) = -D_t\boldsymbol{\omega} - \tfrac{3}{2}\boldsymbol{u}\cdot\boldsymbol{u}D_x\boldsymbol{\omega} + 3\boldsymbol{u}_x\cdot\boldsymbol{\omega}\,\boldsymbol{u} - 3\boldsymbol{u}\cdot\boldsymbol{u}_x\,\boldsymbol{\omega} - D_x^3\boldsymbol{\omega} = 0. \tag{3.22}$$

The vector mKdV equation (3.20) admits the recursion operator

$$\mathcal{R} = D_x^2 + \boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{u}_x D_x^{-1}(\boldsymbol{u}\cdot) - \boldsymbol{u}\lrcorner D_x^{-1}(\boldsymbol{u}_x\wedge) \tag{3.23}$$

where $\wedge$ denotes the antisymmetric outer product of two vectors and $\lrcorner$ denotes the interior product (i.e. contraction) of a vector against a tensor, namely $\boldsymbol{c}\lrcorner(\boldsymbol{a}\wedge\boldsymbol{b}) = (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b} - (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}$. This expression (3.23) is a manifestly $SO(N)$-invariant version of the vector mKdV recursion operator first derived in [10]. The adjoint of $\mathcal{R}$ is given by the similar operator

$$\mathcal{R}^* = D_x^2 + \boldsymbol{u}\lrcorner(\boldsymbol{u}\wedge) + \boldsymbol{u}_x\lrcorner D_x^{-1}(\boldsymbol{u}\wedge) + \boldsymbol{u}D_x^{-1}(\boldsymbol{u}\cdot D_x) \tag{3.24}$$

(closely resembling the form of the sine–Gordon recursion operator in the scalar case $N = 1$, when the $\wedge$ terms vanish). We now consider the infinite sequence of mKdV adjoint-symmetries generated by $\boldsymbol{\omega}_{(k)} = (\mathcal{R}^*)^k\boldsymbol{u}$, $k = 0, 1, 2, \ldots$, starting from the obvious solution of equation (3.22), $\boldsymbol{\omega} = \boldsymbol{u}$. By means of the scaling symmetry

$$\delta\boldsymbol{u} = \boldsymbol{\eta}_{\text{s}} = -\boldsymbol{u} - 3t\boldsymbol{u}_t - x\boldsymbol{u}_x \tag{3.25}$$

the conservation law formula (2.8) yields the conserved density

$$\Phi^t_\omega = -(3t\boldsymbol{u}_t + x\boldsymbol{u}_x + \boldsymbol{u}) \cdot \boldsymbol{\omega} \simeq -D_x^{-1}(\boldsymbol{u} \cdot D_x\boldsymbol{\omega}_{(k)}) = \Phi^t_{(k)}. \tag{3.26}$$

Note this can be expressed in terms of the recursion operator $\mathcal{R}^*$ by

$$\Phi^t_{(k)} = -\boldsymbol{u}_x \cdot \mathcal{R}^*_{\text{nonloc}}(\boldsymbol{\omega}_{(k)})/\boldsymbol{u} \cdot \boldsymbol{u}_x. \tag{3.27}$$

As $\boldsymbol{\omega}_{(k)} \rightarrow \lambda^{-(1+2k)}\boldsymbol{\omega}_{(k)}$ under the mKdV scaling symmetry, we see from the scaling formula (2.12) that each adjoint-symmetry $\boldsymbol{\omega}_{(k)}$ is a multiplier for a local conservation law on mKdV solutions, given by

$$D_t\Psi^t_{(k)}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots) + D_x\Psi^x_{(k)}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots) = 0 \tag{3.28}$$

with

$$\Psi^t_{(k)} \simeq \frac{-1}{2k+1}\Phi^t_{(k)}. \tag{3.29}$$

However, in contrast to the scalar KdV and sine–Gordon cases, here expressions (3.26) and (3.29) do not lead in any immediate way to a recursion operator for the mKdV conservation laws (since $D_x\boldsymbol{\omega}_{(k)}$ cannot be expressed directly in terms of $\Phi^t_{(k)}$, $\boldsymbol{u}$ and their derivatives). Nevertheless we have an explicit recursion formula

$$\Psi^t_{(k)} = \frac{1}{2k+1}D_x^{-1}(\boldsymbol{u} \cdot D_x((\mathcal{R}^*)^k\boldsymbol{u})) \tag{3.30}$$

for generating all of the local conservation laws (3.28) (to within a trivial conserved density $D_x\Theta$).

Other soliton field equations, such as the nonlinear Schrödinger equation, Tzetzeica equation, Harry–Dym equation, Boussinesq equation and their variants [11], as well as more general multi-component scalar/vector field equations [9], can be treated in a similar way to the preceeding examples.

3.2. Fluid flow and wave propagation

Secondly, field equations for fluid flow and nonlinear wave propagation in 2 + 1 and 3 + 1 dimensions will be considered.

3.2.1. Euler equations. The field equations for an incompressible inviscid fluid in two or three spatial dimensions are given by

$$\Upsilon^i(\boldsymbol{u}, \partial_t\boldsymbol{u}, \partial_{\boldsymbol{x}}\boldsymbol{u}, P) = \partial_t u^i + u^j\partial_j u^i + \rho^{-1}\partial^i P = 0 \qquad \Upsilon(\partial_{\boldsymbol{x}}\boldsymbol{u}) = \partial_i u^i = 0 \tag{3.31}$$

for fluid velocity $u^i(t, \boldsymbol{x})$ and pressure $P(t, \boldsymbol{x})$, with constant density ρ. This system is invariant under the family of scalings $t \rightarrow \lambda^p t$, $x^i \rightarrow \lambda x^i$, $u^i \rightarrow \lambda^{1-p}u^i$, $P \rightarrow \lambda^{2-2p}P$, for arbitrary $p = \text{const}$. The fluid symmetries are solutions (η^i, η) of the linearized field equations

$$\mathcal{L}_{\boldsymbol{\Upsilon}}(\boldsymbol{\eta}, \eta)^i = D_t\eta^i + u^j D_j\eta^i + \partial_j u^i\eta^j + \rho^{-1}D^i\eta = 0 \qquad \mathcal{L}_\Upsilon(\boldsymbol{\eta})^i = D_i\eta^i = 0 \tag{3.32}$$

for all $\boldsymbol{u}(t, \boldsymbol{x})$, $P(t, \boldsymbol{x})$ satisfying the Euler equations. The adjoint of equations (3.32) is given by

$$\mathcal{L}^*_{\boldsymbol{\Upsilon}}(\boldsymbol{\omega}, \omega)_i = -D_t\omega_i - u^j D_j\omega_i + \partial_i u^j\omega_j - D_i\omega = 0 \qquad \mathcal{L}^*_\Upsilon(\boldsymbol{\omega}) = -\rho^{-1}D^i\omega_i = 0 \tag{3.33}$$

whose solutions (ω_i, ω) for all $\boldsymbol{u}(t, \boldsymbol{x})$, $P(t, \boldsymbol{x})$ satisfying (3.31) are the adjoint-symmetries of the Euler equations. In addition to scaling symmetries,

$$\delta u^i = {\eta_{\text{s}}}^i = (1-p)u^i - x^j\partial_j u^i - pt\partial_t u^i \tag{3.34}$$

$$\delta P = \eta_{\rm s} = (2-2p)P - x^j\partial_j P - pt\partial_t P \tag{3.35}$$

the Euler equations are well known to possess the Galilean group [1, 12] of symmetries, comprising time translations

$$\delta u^i = \eta^i = \partial_t u^i = -u^j\partial_j u^i - \rho^{-1}\partial^i P \qquad \delta P = \eta = \partial_t P \tag{3.36}$$

Galilean boosts, with velocity $v^i = \text{const}$,

$$\delta u^i = \eta^i = tv^j\partial_j u^i - v^i \qquad \delta P = \eta = tv^j\partial_j P \tag{3.37}$$

and space translations and rotations

$$\delta u^i = \eta^i = \mathcal{L}_\xi u^i = \xi^j\partial_j u^i - (u^j\partial_j\xi^i) \qquad \delta P = \eta = \mathcal{L}_\xi P = \xi^j\partial_j P \tag{3.38}$$

where $\xi^i(x)$ is a Killing vector of the Euclidean space in which the fluid flow takes place, i.e. $\partial^{(i}\xi^{j)} = 0$. (In particular, $\xi^i = a^i = \text{const}$ yields translations, and $\xi^i = b^{ij}x_j, b^{ij} = -b^{ji} = \text{const}$, yields rotations.) Corresponding adjoint-symmetries are given by the relations

$$\omega_i = \partial\eta^j/\partial(\partial^i u^j) \tag{3.39}$$

$$\omega = \int \omega_i\,\mathrm{d}u^i - \partial_t\omega_i\,\mathrm{d}x^i + P\partial\eta^i/\partial(\partial^i P) \tag{3.40}$$

yielding

$$\omega_i = \delta_{ij}u^j \qquad \omega = \tfrac{1}{2}\delta_{ij}u^i u^j + \rho^{-1}P \tag{3.41}$$

$$\omega_i = \delta_{ij}\xi^j \qquad \omega = \delta_{ij}\xi^i u^j \tag{3.42}$$

$$\omega_i = \delta_{ij}tv^j \qquad \omega = \delta_{ij}v^i(tu^j - x^j). \tag{3.43}$$

Now, typically, local conservation laws

$$D_t\Psi^t(\boldsymbol{x},\boldsymbol{u},P) + D_i\Psi^i(\boldsymbol{x},\boldsymbol{u},P) = 0 \tag{3.44}$$

on solutions of the Euler equations are derived through consideration of Newton's laws applied to fluid elements [12] or by Noether's theorem in a Hamiltonian formulation [1, 13]. In contrast, the conservation law formula (2.8) in terms of any adjoint-symmetry (ω_i, ω) directly yields a conserved density

$$\Phi^t_{(\omega,\omega)} = ((1-p)u^i - x^j\partial_j u^i - pt\partial_t u^i)\omega_i \simeq ((ptu^j - x^j)\partial_j u^i + (1-p)u^i)\omega_i \tag{3.45}$$

to within a trivial conserved density $D_i\Theta^i$. Here, this formula easily leads to momentum and angular momentum

$$\Psi^t_\xi = \delta_{ij}\xi^i u^j \simeq \Phi^t_\xi\big/w_\xi \tag{3.46}$$

from the Killing vector adjoint-symmetries (3.42), and Galilean momentum

$$\Psi^t_v = \delta_{ij}tv^i u^j \simeq \Phi^t_v\big/w_v \tag{3.47}$$

from the boost adjoint-symmetry (3.43), as well as energy

$$\Psi^t_u = \tfrac{1}{2}\delta_{ij}u^i u^j \simeq \Phi^t_u\big/w_u \tag{3.48}$$

from the fluid velocity adjoint-symmetry (3.41), to within proportionality factors. (A useful identity in these calculations is $u^j = D_i(x^j u^i)$ on fluid solutions.) From the scaling formula (2.12), in three dimensions, it follows that $w_\xi = 4 - p$ when $\boldsymbol{\xi}$ is a translation, $w_\xi = 5 - p$ when $\boldsymbol{\xi}$ is a rotation, $w_u = 5 - 2p$ and $w_v = 4$, representing the scaling weights of, respectively, the integrals for momentum and angular momentum $\int \boldsymbol{\xi}\cdot\boldsymbol{u}\,\mathrm{d}^3x$, energy $\int \frac{1}{2}|\boldsymbol{u}|^2\,\mathrm{d}^3x$, and Galilean momentum $\int t\boldsymbol{v}\cdot\boldsymbol{u}\,\mathrm{d}^3x$. Note, for the dilation scaling $p = 1$, all the scaling weights are positive and hence the conservation laws (3.46), (3.47), (3.48) are noncritical. These weights decrease by 1 in two dimensions, leading to the same conclusions.

The Euler equations also are known to possess a vorticity conservation law [14], which is unrelated to symmetries in contrast with the energy and momentum conservation laws [13]. Here, a derivation will be given by formula (3.45) directly in terms of fluid adjoint-symmetries.

In three dimensions, vorticity is the curl of the fluid velocity

$$\Omega^i = \epsilon^i{}_{jk}\partial^j u^k \tag{3.49}$$

satisfying the vorticity equations [12]

$$\partial_t\Omega^i + u^j\partial_j\Omega^i - \Omega^j\partial_j u^i = 0 \qquad \partial_i\Omega^i = 0 \tag{3.50}$$

where $\epsilon^i{}_{jk}$ is the cross-product operator (i.e. $\epsilon_{ijk} = \delta_{li}\epsilon^l{}_{jk} = \epsilon_{[ijk]}$ is the totally antisymmetric symbol). We observe these equations have precisely the form of the adjoint-symmetry equations (3.33) and hence

$$\omega_i = \delta_{ij}\Omega^j \qquad \omega = 0 \tag{3.51}$$

yields a corresponding fluid adjoint-symmetry. Then the conserved density formula (3.45) leads to

$$\Psi^t_\Omega = \tfrac{1}{2}\epsilon_{ijk}u^i\partial^j u^k \simeq \Phi^t_\Omega\big/w_\Omega \tag{3.52}$$

where, from scaling formula (2.12), $w_\Omega = 4-2p$ is the scaling weight of the vorticity integral $\int \boldsymbol{u}\cdot(\boldsymbol{\partial}\times\boldsymbol{u})\,\mathrm{d}^3x$. Physically, this conserved quantity describes the total helicity (degree of knottedness) of vortex filaments. Note its scaling weight is noncritical provided $p \neq 2$.

The situation in two dimensions is slightly different. The role of fluid vorticity is played by the scalar curl

$$\Omega = \epsilon_{jk}\partial^j u^k \tag{3.53}$$

where $\epsilon_{jk} = \epsilon_{[jk]}$ is the antisymmetric symbol. This scalar vorticity satisfies the conservation equation

$$\partial_t\Omega + u^j\partial_j\Omega = 0. \tag{3.54}$$

By taking a curl, we immediately see that

$$\omega_i = \epsilon_{ij}\partial^j\Omega \qquad \omega = -\Omega^2/2 \tag{3.55}$$

satisfy the fluid adjoint-symmetry equations (3.33). The conserved density formula (3.45) now yields

$$\Psi^t_\Omega = \tfrac{1}{2}(\epsilon_{jk}\partial^j u^k)^2 \simeq \Phi^t_\Omega\big/w_\Omega \tag{3.56}$$

with the proportionality factor $w_\Omega = 2-2p$ given by the scaling weight of the conserved vorticity integral $\int(\boldsymbol{\partial}\times\boldsymbol{u})^2\,\mathrm{d}^2x$ (where $\boldsymbol{\partial}\times\boldsymbol{u}$ denotes the scalar curl $\epsilon_{jk}\partial^j u^k$). Note that this vorticity quantity is noncritical if $p \neq 1$, i.e. other than for a dilation scaling. More generally, any function $f(\Omega)$ in two dimensions is also a conserved density due to conservation (3.54) of the vorticity $\Omega = \boldsymbol{\partial}\times\boldsymbol{u}$, but the resulting vorticity integral $\int f(\boldsymbol{\partial}\times\boldsymbol{u})\,\mathrm{d}^2x$ will have a well-defined scaling weight only if $p = 0$, namely for a spatial dilation scaling. In this case the conserved density

$$\Psi^t_f = f(\epsilon_{jk}\partial^j u^k) \simeq \Phi^t_f\big/w_f \tag{3.57}$$

again arises directly from formula (3.45), through the adjoint-symmetry

$$\omega_i = \epsilon_{ij}\partial^j\Omega f'' \qquad \omega = f - \Omega f'. \tag{3.58}$$

Here the proportionality factor is simply $w_f = 2$, due to the scaling invariance of $f(\Omega)$, and consequently the vorticity integral is noncritical. The same conclusion holds for the vorticity integral in three dimensions if a spatial dilation scaling, $p = 0$, is considered.

The Navier–Stokes equations and polytropic gas dynamics equations can be treated analogously.

3.2.2. Nonlinear wave equation. The scalar field equation for $u(t, \boldsymbol{x})$

$$\Upsilon(u, \partial_t u, \partial_{\boldsymbol{x}} u) = \partial_t^2 u - \partial^i \partial_i u \pm u^\upsilon = -g^{\alpha\beta}\partial_\alpha \partial_\beta u \pm u^\upsilon = 0 \tag{3.59}$$

describes a nonlinear wave with interaction strength depending on a positive integer $\upsilon > 1$ (where $g_{\alpha\beta}$ is the Minkowski metric tensor and $x^\alpha = (t, \boldsymbol{x})$ are the Minkowski spacetime coordinates). This is a Lagrangian wave equation invariant under the scaling $x^\alpha \to \lambda x^\alpha$, $u \to \lambda^q u$, for $q = 2/(1-\upsilon) \neq 0$. The corresponding scaling symmetry

$$\delta u = \eta_{\mathrm{s}} = qu - x^\alpha \partial_\alpha u \tag{3.60}$$

is a solution of the linearized field equation

$$\mathcal{L}_\Upsilon(\eta) = -g^{\alpha\beta} D_\alpha D_\beta u \pm \upsilon u^{\upsilon-1}\eta = 0 \tag{3.61}$$

for all $u(t, \boldsymbol{x})$ satisfying the wave equation (3.59). Here, symmetries are the same as adjoint-symmetries, since $\mathcal{L}^*_\Upsilon = \mathcal{L}_\Upsilon$. The additional spacetime symmetries (Poincaré group) of the wave equation (3.59) for arbitrary $\upsilon > 0$ are given by translations, rotations and boosts,

$$\delta u = \eta = \mathcal{L}_\xi u = \xi^\alpha \partial_\alpha u \tag{3.62}$$

where $\xi^\alpha(x)$ is the Killing vector of Minkowski space, $\partial^{(\alpha}\xi^{\beta)} = 0$. For certain interaction powers, $\upsilon = 5$ in 2 + 1 dimensions and $\upsilon = 3$ in 3 + 1 dimensions, the wave equation (3.59) also admits [15] inversion symmetries

$$\delta u = \eta = \mathcal{L}_\xi u + \tfrac{1}{6}\operatorname{div}\xi\, u \tag{3.63}$$

(where $\operatorname{div}\xi = \partial_\alpha \xi^\alpha$) associated with conformal Killing vectors $\xi^\alpha = c^\beta x_\beta x^\alpha - \frac{1}{2}c^\alpha x_\beta x^\beta$, $c^\beta = \mathrm{const}$, satisfying $\partial^{(\alpha}\xi^{\beta)} = \Omega g^{\alpha\beta}$ for a conformal factor $\Omega(x)$. Through Noether's theorem, all spacetime symmetries (3.62) and (3.63) are known to be multipliers [15] for local conservation laws on solutions of the wave equation (3.59),

$$D_\alpha \Psi^\alpha_\xi(t, \boldsymbol{x}, u, \partial_t u, \partial_{\boldsymbol{x}} u) = 0 \tag{3.64}$$

with

$$\Psi^\alpha_\xi = -T^\alpha{}_\beta(u, \partial u)\xi^\beta - \frac{1}{\upsilon+1}\left(\operatorname{div}\xi\, u \partial^\alpha u - \frac{1}{2}(\partial^\alpha \operatorname{div}\xi)u^2\right) \tag{3.65}$$

given in terms of the conserved stress–energy tensor

$$T^\alpha{}_\beta(u, \partial u) = \partial^\alpha u \partial_\beta u - \frac{1}{2}\delta^\alpha_\beta\left(\partial^\gamma u \partial_\gamma u \pm \frac{2}{\upsilon+1}u^{\upsilon+1}\right). \tag{3.66}$$

The conservation $\partial_\alpha T^\alpha{}_\beta(u, \partial u) = 0$ of this tensor (3.66) on solutions of the wave equation provides a well-known alternative derivation of the conservation laws associated with spacetime Killing vectors,

$$\Psi^\alpha_\xi = -T^\alpha{}_\beta(u, \partial u)\xi^\beta \qquad \text{for} \quad \partial^{(\alpha}\xi^{\beta)} = 0. \tag{3.67}$$

Here, the resulting conserved densities instead will be obtained from the conservation law formula (2.8) directly in terms of the corresponding symmetries (3.62) and (3.63). This yields

$$\begin{aligned}\Phi^\alpha_\eta &= ((q-1)\partial^\alpha u - x^\beta \partial^\alpha \partial_\beta u)\left(\xi^\gamma \partial_\gamma u + \frac{1}{\upsilon+1}\operatorname{div}\xi\, u\right)\\ &\quad - \left(qu - x^\beta \partial_\beta u\right)\left(\partial^{[\alpha}\xi^{\gamma]}\partial_\gamma u + \xi^\gamma \partial^\alpha \partial_\gamma u + \frac{1}{2}\operatorname{div}\xi\, \partial^\alpha u + \frac{1}{\upsilon+1}(\partial^\alpha \operatorname{div}\xi)u\right)\\ &\simeq -w_\xi \Psi^\alpha_\xi \end{aligned} \tag{3.68}$$

to within a trivial conserved density ($D_\beta \Theta^{\alpha\beta}$, with $\Theta^{\alpha\beta} = -\Theta^{\beta\alpha}$). The proportionality factor w_ξ is, by the scaling formula (2.12), the scaling weight of the flux integrals $\int \Psi^\alpha_\xi n_\alpha \,\mathrm{d}\Sigma$, on a

$t = \text{const}$ spatial hypersurface Σ with normal vector $n_\alpha = \partial_\alpha t$. For translations $\xi^\alpha = a^\alpha =$ const, we have

$$w_\xi = 4/(1 - \upsilon) \tag{3.69}$$

in 2 + 1 dimensions, while in 3 + 1 dimensions,

$$w_\xi = (5 - \upsilon)/(1 - \upsilon). \tag{3.70}$$

The weight w_ξ increases by 1 for rotations and boosts $\xi^\alpha = b^{\alpha\beta}x_\beta$, $b^{\alpha\beta} = -b^{\beta\alpha} =$ const, and increases by 1 again for inversions (3.63) so thus $w_\xi = 1$ in the case of proper conformal Killing vectors. Hence, a critical case $w_\xi = 0$ only occurs for translations in 3 + 1 dimensions when $\upsilon = 5$, and for rotations and boosts in 3 + 1 dimensions when $\upsilon = 3$ as well as in 2 + 1 dimensions when $\upsilon = 5$, corresponding to scaling invariance of the energy–momentum integral $\int -T^\alpha{}_\beta(u, \partial u)a^\beta n_\alpha \, \mathrm{d}\Sigma$ and of the angular-boost momentum integral $\int -T^\alpha{}_\beta(u, \partial u)b^{\beta\gamma}x_\gamma n_\alpha \, \mathrm{d}\Sigma$ in these cases. Consequently, $\Phi^\alpha_\eta \simeq 0$ is trivial only for these critical interaction powers[7] and Killing vectors. In this situation, all local conservation laws are produced nevertheless from the more general formula (2.5) directly in terms of pairs of Killing vector symmetries (3.62),

$$\Psi^\alpha(\eta_1, \eta_2) \simeq -T^\alpha{}_\beta(u, \partial u)\xi^\beta \tag{3.71}$$

where $\xi^\alpha = [\xi_1, \xi_2]^\alpha$ is the commutator of the Killing vectors. The same result holds even for the noncritical cases where $w_\xi \neq 0$.

Other nonlinear wave equations, such as sigma models and wavemap equations [16], can be treated in the same way.

3.3. Gauge theories

Finally, Yang–Mills fields and gravitational fields in 3 + 1 dimensions will be considered.

3.3.1. Yang–Mills theory. The Yang–Mills field on the Minkowski space $(\mathbb{R}^4, g_{\alpha\beta})$ is a vector potential $A^a_\alpha(x)$ that takes values in an internal Lie algebra $\mathcal{G} = \left(\mathbb{R}^N, c^a{}_{bc}\right)$. Associated with A^a_α is the Yang–Mills covariant derivative

$$\nabla^A_\alpha = \partial_\alpha + c^a{}_{bc}A^b_\alpha \tag{3.72}$$

and the Yang–Mills field strength tensor

$$F^a_{\alpha\beta} = \partial_{[\alpha}A^a_{\beta]} + \tfrac{1}{2}c^a{}_{bc}A^b_\alpha A^c_\beta \tag{3.73}$$

where $c^a{}_{bc}$ denotes the structure constants of the Lie algebra $\mathcal{G}$. The Yang–Mills equation with gauge group based on $\mathcal{G}$ is then given by

$$\Upsilon^a_\mu(A, \partial A, \partial^2 A) = g^{\alpha\beta}\nabla^A_\alpha F^a_{\beta\mu} = 0 \tag{3.74}$$

which is invariant under the scaling $x^\alpha \to \lambda x^\alpha$, $A^a_\alpha \to \lambda^{-1}A^a_\alpha$. Whenever the gauge group is semisimple, the Yang–Mills equation (3.74) arises from a Lagrangian (see, e.g., [17]), and in this situation both the symmetries and adjoint-symmetries of this field equation (3.74) are given by solutions $\delta A^a_\alpha = \eta^a_\alpha$ of the linearized Yang–Mills equation

$$\mathcal{L}_\Upsilon(\eta)^a_\mu = g^{\alpha\beta}\left(\nabla^A_\alpha\nabla^A_{[\beta}\eta^a_{\mu]} + c^a{}_{bc}\eta^b_\alpha F^c_{\beta\mu}\right) = 0 \tag{3.75}$$

[7] The interaction power for which the energy conservation law has critical scaling weight coincides with the notion of the critical power for blow-up to occur for solutions with initial-data of large energy. See, e.g., [16].

for all Yang–Mills solutions $A^a_\alpha(x)$. Note, here, $\nabla^A_\alpha = D_\alpha + c^a{}_{bc}A^b_\alpha$ acts as a total derivative operator. The well-known local symmetries of the Yang–Mills equation (3.74) are comprised by gauge symmetries

$$\delta A^a_\alpha = \nabla^A_\alpha \chi^a \tag{3.76}$$

involving any Lie-algebra valued scalar function $\chi^a(x, A, \partial A, \ldots)$, and spacetime symmetries

$$\delta A^a_\alpha = 2\xi^\beta F^a_{\beta\alpha} = \mathcal{L}_\xi A^a_\alpha - \nabla^A_\alpha\big(\xi^\beta A^a_\beta\big) \tag{3.77}$$

where $\xi^\beta(x)$ is any conformal Killing vector on Minkowski space, $\partial^{(\alpha}\xi^{\beta)} = \frac{1}{4}g^{\alpha\beta}\,\mathrm{div}\,\xi$. In the case of a dilation Killing vector, $\xi^\beta = x^\beta$, the spacetime symmetry (3.77) reduces to a sum of the Yang–Mills scaling symmetry and a gauge symmetry,

$$\delta_{\rm s} A^a_\alpha = \eta_{\rm s}{}^a_\alpha = -x^\beta\partial_\beta A^a_\alpha - A^a_\alpha = -2x^\beta F^a_{\beta\alpha} - \nabla^A_\alpha\big(x^\beta A^a_\beta\big). \tag{3.78}$$

A recent classification analysis [18] has proved that these are in fact the only nontrivial local symmetries admitted by the Yang–Mills equation if the Lie algebra $\mathcal{G}$ is real and simple. However, for a simple Lie algebra $\mathcal{G}$ with a complex structure, the same analysis found that the Yang–Mills equation also admits complexified spacetime symmetries

$$\delta A^a_\alpha = 2j^a{}_b\xi^\beta F^b_{\beta\alpha} \tag{3.79}$$

where $j^a{}_b$ is the complex structure map on $\mathcal{G}$, satisfying the properties

$$j^a{}_b j^b{}_c = -\delta^c_a \qquad j^a{}_b c^b{}_{cd} = c^a{}_{ed} j^e{}_c \qquad k_{a[b} j^a{}_{c]} = 0 \tag{3.80}$$

with $k_{ab} = c^c{}_{ad}c^d{}_{bc}$ being the Cartan–Killing metric on $\mathcal{G}$. Through Noether's theorem, these symmetries (3.76) to (3.79) are multipliers for local conservation laws $D_\alpha\Psi^\alpha(x, A, \partial A) = 0$ consisting of, respectively,

$$\Psi^\alpha = g^{\alpha\nu}g^{\beta\gamma}k_{ab}F^a_{\nu\beta}\nabla^A_\gamma\chi^b \simeq 0 \tag{3.81}$$

related to the Bianchi identity on $F^a_{\alpha\beta}$, and

$$\Psi^\alpha = T^\alpha{}_\beta(F)\xi^\beta \qquad \Psi^\alpha = \tilde{T}^\alpha{}_\beta(F)\xi^\beta \tag{3.82}$$

given by the conserved Yang–Mills stress–energy tensor

$$T_{\alpha\beta}(F) = g^{\nu\sigma}k_{ab}\big(F^a_{\alpha\nu}F^b_{\beta\sigma} - \tfrac{1}{4}g_{\alpha\beta}g^{\mu\gamma}F^a_{\mu\nu}F^b_{\gamma\sigma}\big) \tag{3.83}$$

and its complexification

$$\tilde{T}_{\alpha\beta}(F) = g^{\nu\sigma}j_{ab}\big(F^a_{\alpha\nu}F^b_{\beta\sigma} - \tfrac{1}{4}g_{\alpha\beta}g^{\mu\gamma}F^a_{\mu\nu}F^b_{\gamma\sigma}\big). \tag{3.84}$$

These conservation laws (3.82) yield (complexified) energy–momentum, angular and boost momentum for translation, rotation and boost Killing vectors $\partial^{(\alpha}\xi^{\beta)} = 0$, and additional quantities for dilation and inversion Killing vectors $\partial^{(\alpha}\xi^{\beta)} = \frac{1}{4}\,\mathrm{div}\,\xi\, g^{\alpha\beta} \neq 0$.

However, the previous results do not fully settle the classification of local conservation laws of the Yang–Mills equation (3.74), since it leaves open the question of whether any trivial symmetries could yield nontrivial conservation laws by Noether's theorem. For Lagrangian field equations whose principal part (i.e. the highest derivative terms) is nondegenerate, it is known that there is a one-to-one correspondence between nontrivial variational symmetries and nontrivial conservation laws [1, 4]. But this correspondence is not automatic for a field equation with gauge symmetries, due to the resulting degeneracy of the field equation's principal part, in contrast to the previous examples in this section. Here, through an application of formula (2.8), the gap in the classification of Yang–Mills conservation laws will be addressed.

We consider local symmetries

$$\delta A^a_\alpha = \eta^a_\alpha(x, A, \partial A, \ldots) \tag{3.85}$$

assumed to be homogeneous with respect to the Yang–Mills scaling (3.78),

$$\delta_{\rm s}\eta^a_\alpha = r\eta^a_\alpha - x^\beta\partial_\beta\eta^a_\alpha \tag{3.86}$$

with scaling weight $r = \text{const}$. As noted in section 2, there is no loss of generality in such a homogeneity restriction. Now, formula (2.8) yields a conserved current generated from any such symmetry,

$$\Phi^\alpha(\eta, \delta_{\rm s}A) \simeq 2k^{ab}\big(x^\mu F^a_{\mu\beta}g^{\beta[\nu}\nabla^{A\alpha]}\eta^b_\nu - \eta^a_\nu\big(x^\mu\nabla^A_\mu F^{b\alpha\nu} + F^{b\alpha\nu}\big)\big) \tag{3.87}$$

with the current being linear and homogeneous in η^a_α and $\nabla^A_\mu\eta^a_\alpha$. If $\eta^a_\alpha = Q^a_\alpha$ is a multiplier for a local conservation law of the Yang–Mills equation (3.74),

$$k_{ab}g^{\alpha\beta}Q^a_\alpha\nabla^{A\nu}F^b_{\nu\beta} = D_\alpha\Psi^\alpha_Q \tag{3.88}$$

where the current

$$\Psi^\alpha_Q(x, A, \partial A, \ldots) \tag{3.89}$$

can be assumed homogeneous under the Yang–Mills scaling (3.78), then the scaling formula (2.12) gives the relation

$$\Phi^\alpha(\eta, \delta_{\rm s}A) \simeq (r+1)\Psi^\alpha_Q \tag{3.90}$$

to within a trivial conserved current. Hence, for a variational symmetry that is trivial, so $\eta^a_\alpha = 0$ for solutions of (3.74), we see that, if $r \neq -1$,

$$\Psi^\alpha_Q \simeq \frac{1}{r+1}\Phi^\alpha(0, \delta_{\rm s}A) = 0 \tag{3.91}$$

is a trivial noncritical current on Yang–Mills solutions. Note the factor $r + 1$ here is precisely the scaling weight of the flux integral of the current Ψ^α_Q. This establishes the following classification result.

Proposition 3.1. *No nontrivial conservation laws (3.88) that are noncritical—i.e. whose associated conserved quantity* $\int \Psi^\alpha\partial_\alpha t\,{\rm d}\Sigma$, *on a spatial hypersurface* Σ *given by* $t = \text{const}$, *has non-zero scaling weight—may arise from variational trivial symmetries (3.85) and (3.86) of the Yang–Mills equation (3.74).*

3.3.2. General relativity. The vacuum gravitational field equation [19] on a four-dimensional spacetime manifold is given by

$$\Upsilon^{\alpha\beta}(g, \partial g, \partial^2 g) = G^{\alpha\beta} = 0 \tag{3.92}$$

where $G_{\alpha\beta}$ is the Einstein tensor (i.e. trace-reversed Ricci tensor $R_{\alpha\beta}$) for the spacetime metric $g_{\alpha\beta}(x)$. Its linearized field equation is given by

$$\mathcal{L}_\Upsilon(\eta)^{\alpha\beta} = -\tfrac{1}{2}\big(\nabla^{g\mu}\nabla^g_\mu\bar\eta^{\alpha\beta} - 2\nabla^g_\mu\nabla^{g(\alpha}\bar\eta^{\beta)\mu} + g^{\alpha\beta}\nabla^g_\mu\nabla^g_\nu\bar\eta^{\mu\nu}\big) = 0 \tag{3.93}$$

whose solutions for all $g_{\alpha\beta}(x)$ satisfying the Einstein equation (3.92) are the symmetries $\delta g^{\alpha\beta} = \eta^{\alpha\beta}(x, g, \partial g, \ldots)$ of the gravitational field, where ∇^g_μ is the covariant total derivative operator associated with the metric, $\nabla^g_\mu g_{\alpha\beta} = 0$, and 'bar' denotes trace-reversal on a given tensor. Here, since the gravitational field equation (3.92) comes from a Lagrangian [19], adjoint-symmetries are the same as symmetries. It is known that the only admitted nontrivial local symmetries [20] consist of a constant conformal scaling

$$\delta g_{\alpha\beta} = g_{\alpha\beta} \tag{3.94}$$

and diffeomorphism gauge symmetries

$$\delta g_{\alpha\beta} = \mathcal{L}_\zeta g_{\alpha\beta} = 2\nabla^g_{(\alpha}\zeta_{\beta)} \tag{3.95}$$

for any local vector field $\zeta^\beta(x, g, \partial g, \ldots)$. If we consider local conservation laws of the gravitational field equation (3.92),

$$\nabla^g_\alpha \Psi^\alpha(x, g, \partial g, \ldots) = 0 \tag{3.96}$$

then through Noether's theorem the constant conformal scaling (3.94) is not a multiplier, while the diffeomorphisms (3.95) yield a trivial conservation law $\Psi^\alpha = \zeta^\beta G_\beta{}^\alpha = 0$ on solutions of (3.92).

The results in [20] assert that in fact there exist no nontrivial local conservation laws (3.96), but does not provide a full statement of the proof. Here a complete classification proof will be given for diffeomorphism–covariant conservation laws, by use of a covariant version of the conservation law formula (2.8). A suitable dilation scaling symmetry is provided by the diffeomorphism symmetry (3.95) specialized to a homothetic vector field $\zeta^\alpha = \xi^\alpha$, namely

$$\delta_s g^{\alpha\beta} = \mathcal{L}_\xi g \qquad \nabla^g_\alpha \xi^\alpha = \text{const} \neq 0. \tag{3.97}$$

Now, a local conservation law (3.96) is diffeomorphism–covariant whenever it is of the form

$$\nabla^g_\alpha \Psi^\alpha(g, R, \nabla^g R, \ldots) = 0 \tag{3.98}$$

on solutions of the Einstein equation (3.92), with Ψ^α satisfying the natural transformation property $\delta\Psi^\alpha = \mathcal{L}_\zeta \Psi^\alpha$ under all diffeomorphism symmetries (3.95). Correspondingly, without loss of generality we consider diffeomorphism–covariant local symmetries

$$\delta g^{\alpha\beta} = \eta^{\alpha\beta}(g, R, \nabla^g R, \ldots) \tag{3.99}$$

which are necessarily homogeneous with respect to the dilation scaling

$$\delta_s \eta^{\alpha\beta} = \mathcal{L}_\xi \eta^{\alpha\beta}. \tag{3.100}$$

Then for any such symmetry (3.99), formula (2.8) yields a conserved current

$$\Phi^\alpha(\eta, \mathcal{L}_\xi g) = -\left(\xi^\nu R_{\nu\mu\beta}{}^\alpha - \nabla^g_\mu \nabla^g_\beta \xi^\alpha\right)\bar{\eta}^{\mu\beta} + \nabla^g_\mu \xi_\beta \nabla^{g\alpha} \bar{\eta}^{\mu\beta} + \nabla^g_\mu \xi^\mu \nabla^g_\nu \bar{\eta}^{\alpha\nu} \tag{3.101}$$

whose dependence on $\eta^{\alpha\beta}$ and $\nabla^g_\mu \eta^{\alpha\beta}$ is linear, homogeneous. It now follows from the scaling formula (2.12) in covariant form that if $\eta^{\alpha\beta} = Q^{\alpha\beta}$ is a multiplier for a local conserved current, $Q_{\alpha\beta} G^{\alpha\beta} = \nabla^g_\alpha \Psi^\alpha_Q$, then to within a trivial conserved current,

$$\Phi^\alpha(\eta, \mathcal{L}_\xi g) \simeq w \Psi^\alpha_Q \qquad w = \nabla^g_\beta \xi^\beta \neq 0. \tag{3.102}$$

Hence, for a trivial multiplier that is diffeomorphism–covariant (3.99), we have

$$\Psi^\alpha_Q \simeq w^{-1} \Phi^\alpha(0, \mathcal{L}_\xi g) = 0 \tag{3.103}$$

on solutions of the Einstein equation (3.92), since $\eta^{\alpha\beta} = 0$. Therefore, the following result holds.

Proposition 3.2. *No nontrivial diffeomorphism–covariant conservation laws (3.98) may arise from variational trivial symmetries (3.99) of the gravitational field equation (3.92).*

Solutions of the Einstein equation (3.92) with a Killing vector $\mathcal{L}_\xi g_{\alpha\beta} = 2\nabla^g_{(\alpha}\xi_{\beta)} = 0$ are known to possess an unexpected extra symmetry [21]

$$\delta g_{\alpha\beta} = \upsilon g_{\alpha\beta} - 2\xi_{(\alpha}\chi_{\beta)} \qquad \delta\xi_\alpha = -\chi_\alpha \xi^\beta \xi_\beta \tag{3.104}$$

where υ is the scalar twist and χ_β is the dual of ξ^β, defined by the equations

$$\nabla^g_\alpha \upsilon = \epsilon_{\alpha\beta\mu\nu}\xi^\beta \nabla^{g\mu}\xi^\nu \qquad 2\nabla^g_{[\alpha}\chi_{\beta]} = \epsilon_{\alpha\beta\mu\nu}\nabla^{g\mu}\xi^\nu. \tag{3.105}$$

Note this symmetry (3.104) is nonlocal due to its dependence on υ, χ_β (in terms of $g_{\alpha\beta}, \xi_\alpha$). Here, a corresponding nonlocal conservation law will be derived through the conservation law formula (2.8) in covariant form using the constant conformal scaling (3.94),

$$\Phi^\alpha(\eta, g) = g_{\nu\mu}\nabla^{g\alpha}\eta^{\nu\mu} - \nabla^g_\mu \eta^{\alpha\mu}. \tag{3.106}$$

This yields, from (3.104),

$$\Phi^\alpha(\eta, g) = \nabla^{g\alpha}\upsilon + 2\nabla^g_\beta(\xi^{(\beta}\chi^{\alpha)}) \simeq 2\xi_\beta \nabla^{g(\beta}\chi^{\alpha)}. \tag{3.107}$$

Moreover, on solutions of the Einstein equation (3.92) with two commuting Killing vectors ξ^α_1 and ξ^α_2, the symmetries (3.104) are known to generate an infinite-dimensional algebra of nonlocal symmetries (the Geroch group [22]). Then the covariant formula (3.106) leads to a corresponding infinite sequence of conservation laws, related to the complete integrability of the system $G_{\alpha\beta} = 0$, $\mathcal{L}_{\xi_1} g_{\alpha\beta} = \mathcal{L}_{\xi_2} g_{\alpha\beta} = 0$.

A similar classification treatment of local and nonlocal conservation laws of the self-dual Einstein equation and self-dual Yang–Mills equation will be given elsewhere.

4. Conclusion

The conservation law expression derived in proposition 2.1 is a generalization of a formula mentioned in [1] in the case of linear PDEs and extends a similar formula considered for self-adjoint PDEs in [5]. A variant of this expression has been central to a recently obtained classification of local conserved currents for linear massless spinorial field equations of spin $s > 0$ [23, 24]. Numerous applications to nonlinear ODEs are presented in [2].

Apart from its main use in generating all noncritical local conservation laws in an algebraic manner—by-passing the standard homotopy integral formula and (adjoint-) invariance conditions—in terms of local (adjoint-) symmetries admitted by any given PDEs, the conservation law expression is able to produce nonlocal conservation laws from any admitted nonlocal (adjoint-) symmetries. Such examples and applications will be considered in a forthcoming paper.

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